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Generalized plasma-like permittivity and thermal Casimir force between real metals

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Abstract

The physical reasons why the Drude dielectric function is not compatible with the Lifshitz formula, as opposed to the generalized plasma-like permittivity, are presented. Essentially, the problem is connected with the finite size of metal plates. It is shown that the Lifshitz theory combined with the generalized plasma-like permittivity is thermodynamically consistent.

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1. Introduction

In the last few years the Casimir effect [1] received common recognition as one of the most important subjects of interdisciplinary interest. The Casimir force is of the same nature as other one-loop vacuum effects of quantum electrodynamics [2]. It arises due to the alteration of the spectrum of electromagnetic zero-point oscillations by material boundaries. Early stages of modern experiments and related theory are reflected in [3]. Recent trends go toward complex experimental and theoretical studies of the Casimir effect, including the applications to nanotechnology. For this purpose, many classical theoretical results on the subject obtained within the framework of quantum field theory (see, e.g., monographs [4–6], reviews [3, 7, 8], proceedings [9] and more recent papers [10, 11]) should be adapted to the case of real material bodies. Realistic material properties are important also for applications of the Casimir effect in nanotechnology [12–14].

The basic theory of the van der Waals and Casimir forces between dielectric materials was proposed by Lifshitz [15]. However, the application of this theory to Drude metals and semiconductors with sufficiently low charge carrier density met serious problems. Namely, at first, it was shown [16, 17] that for Drude metals with perfect crystal lattices the Lifshitz theory violates the Nernst heat theorem. This is connected with the fact that the reflection coefficient for the transverse electric mode of the electromagnetic field at zero frequency is equal to zero if the dielectric permittivity behaves as ω^{-1} when the frequency ω vanishes. For metals with

impurities the Nernst heat theorem is formally preserved [18]. This inclined the proponents of the Drude model to believe that it is applicable together with the Lifshitz theory (different arguments on this problem can be found in [19–22]). However, precision measurements of the Casimir force [23–26] excluded the Drude model at a 99.9% confidence level.

Another problem arises when the Lifshitz theory is applied to dielectrics or semiconductors with not too high density of free charge carriers. In this case, the Nernst heat theorem is violated if the conductivity at zero frequency is taken into account [27–30]. Here, it is the discontinuity of the transverse magnetic mode at zero frequency, which is responsible for that violation. Furthermore, it was demonstrated [31, 32] that the inclusion of the conductivity at zero frequency into the Lifshitz theory leads to a contradiction with experiment. Until recently there exists not any theoretical approach to the thermal Casimir force which would be in agreement with both short separation experiments [33, 34] and longer separation experiments [23–26]. The approach using the usual, nondissipative, plasma model was shown to be in agreement with longer separation experiment [23–26], but to be in contradiction with the experiment performed at short separations [33, 34]. The impedance approach [35] was also found in agreement with longer separation experiments [23–26], but it is simply not applicable at shorter separations characteristic for the experiment of [33, 34]. Because of this, the Lifshitz theory at zero temperature by necessity was used for the comparison between the short separation experiment [33, 34] and theory, even though that experiment was performed at a room temperature of 300 K.

Recently, a new theoretical approach to the thermal Casimir force between real metals has been proposed [36] using the generalized plasma-like dielectric permittivity. The latter includes dissipation processes due to the interband transitions of core electrons but disregards dissipation due to scattering processes of free electrons. As was shown in [36], the Lifshitz formula combined with the generalized plasma-like dielectric permittivity is consistent with both short and long separation experiments. It also exactly satisfies the Kramers–Kronig relations. However, the question why one should include one type of dissipation (interband transitions of core electrons) to fit theory to experiment while disregarding another one (scattering processes of free electrons) remained unresolved.

In this paper, we present and discuss the physical explanation why the Drude dielectric function cannot be used to describe the thermal Casimir force between metal plates of a *finite area*. The idea of that explanation was briefly published first by Parsegian [37], but did not attract the attention which it deserves. As we show below, the Drude dielectric function is not compatible with the zero-frequency term of the Lifshitz formula if the area of plates is finite. We also perform a rigorous analytical proof of the fact that the Casimir entropy calculated using the generalized plasma-like dielectric permittivity satisfies the Nernst heat theorem.

The paper is organized as follows. In section 2, we explain why the Drude dielectric function is incompatible with the Lifshitz formula in the case of two parallel metallic plates of a finite area. Section 3 contains an asymptotic derivation of the analytic expression for the Casimir entropy in the limit of low temperatures. Here we present a proof for the validity of the Nernst heat theorem in the Lifshitz theory combined with the generalized plasma-like dielectric permittivity. Section 4 contains our conclusions and a discussion.

2. Why the Drude dielectric function is not compatible with the Lifshitz formula for metallic plates of a finite area

In the framework of the Lifshitz theory, the free energy of the fluctuating electromagnetic field between two electrically neutral plane parallel plates of thickness d at temperature T in

thermal equilibrium is given by [3, 15]

$$\mathcal{F}(a, T) = \frac{k_B T}{2\pi} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2} \delta_{0l}\right) \int_0^{\infty} k_{\perp} dk_{\perp} \left\{ \ln [1 - r_{\text{TM}}^2(\xi_l, k_{\perp}) e^{-2aq_l}] + \ln [1 - r_{\text{TE}}^2(\xi_l, k_{\perp}) e^{-2aq_l}] \right\}. \quad (1)$$

Here a is the separation distance between the plates, k_B is the Boltzmann constant, $\xi_l = 2\pi k_B T l / \hbar$ are the Matsubara frequencies defined for any $l = 0, 1, 2, \dots$, and $k_{\perp} = |\mathbf{k}_{\perp}|$ is the magnitude of the wave vector projection onto the plane of the plates. The reflection coefficients for the two independent polarizations of the electromagnetic field (transverse magnetic, TM, and transverse electric, TE) are expressed [38] in terms of the frequency-dependent dielectric permittivity, $\varepsilon(\omega)$, along the imaginary frequency axis:

$$r_{\text{TM}}(\xi_l, k_{\perp}) = \frac{\varepsilon_l^2 q_l^2 - k_l^2}{\varepsilon_l^2 q_l^2 + k_l^2 + 2q_l k_l \varepsilon_l \coth(k_l d)}, \quad (2)$$

$$r_{\text{TE}}(\xi_l, k_{\perp}) = \frac{k_l^2 - q_l^2}{q_l^2 + k_l^2 + 2q_l k_l \coth(k_l d)},$$

where

$$q_l = \sqrt{k_{\perp}^2 + \frac{\xi_l^2}{c^2}}, \quad k_l = \sqrt{k_{\perp}^2 + \varepsilon_l \frac{\xi_l^2}{c^2}}, \quad \varepsilon_l = \varepsilon(i\xi_l). \quad (3)$$

Equation (1) was originally derived [15] for dielectric plates of an infinite area. However, it is commonly used for plates of the finite area S under the condition $a \ll \sqrt{S}$. If this condition is satisfied, corrections to equation (1) due to the finiteness of the plate area are shown to be negligibly small [3, 39] for both dielectric and ideal metal plates. Below we show that this is not the case for metal plates described by the Drude dielectric function where the presence of a real current of conduction electrons leads to a crucially new physical situation.

Papers [18, 21, 22, 40] describe metallic plates by using the dielectric permittivity of the Drude model,

$$\varepsilon_D(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}, \quad (4)$$

where ω_p is the plasma frequency and γ is the relaxation parameter. As is correctly stated by Parsegian (see [37, p 254]), ‘this is valid only in the case of an effectively infinite medium where no walls limit the flow of charges.’ To gain a better understanding of this statement, we derive equation (4) starting from Maxwell equations in an unbounded nonmagnetic metallic medium

$$\begin{aligned} \text{rot } \mathbf{B} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \sigma_0 \mathbf{E}, & \text{div } \mathbf{B} &= 0, \\ \text{rot } \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, & \text{div } \mathbf{E} &= 0. \end{aligned} \quad (5)$$

Here, the electric current density $\mathbf{j} = \sigma_0 \mathbf{E}$ is induced in a metal under the influence of external sources, and σ_0 is the conductivity at zero frequency. Physically the demand that the medium is unbounded means that it should be much larger than the extension of the wave fronts of electromagnetic waves coming from external sources (i.e., of zero-point oscillations and thermal photons).

Solutions of equation (5) can be found in the form of monochromatic waves,

$$\mathbf{E} = \text{Re}[\mathbf{E}_0(\mathbf{r}) e^{-i\omega t}], \quad \mathbf{B} = \text{Re}[\mathbf{B}_0(\mathbf{r}) e^{-i\omega t}], \quad (6)$$

where $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$ satisfy equations

$$\Delta \mathbf{E}_0(\mathbf{r}) + k^2 \mathbf{E}_0(\mathbf{r}) = 0, \quad \Delta \mathbf{B}_0(\mathbf{r}) + k^2 \mathbf{B}_0(\mathbf{r}) = 0, \quad (7)$$

following from (5) with

$$k^2 = \frac{\omega^2}{c^2} + i \frac{4\pi\sigma_0\omega}{c^2} \equiv \frac{\varepsilon_n(\omega)\omega^2}{c^2}. \quad (8)$$

Here the dielectric permittivity of the normal skin effect, $\varepsilon_n(\omega)$, is introduced

$$\varepsilon_n(\omega) = 1 + i \frac{4\pi\sigma_0}{\omega}. \quad (9)$$

This equation is applicable at not too high frequencies (the region of the normal skin effect) where the relation $\mathbf{j} = \sigma_0 \mathbf{E}$ is valid. The Drude model extends the applicability of (9) to higher frequencies, up to the plasma frequency, by making the following replacement in equation (9):

$$\sigma_0 \rightarrow \sigma(\omega) = \frac{\sigma_0 \left(1 + \frac{i\omega}{\gamma}\right)}{1 + \frac{\omega^2}{\gamma^2}}. \quad (10)$$

Substituting (10) in (9) and taking into account that $\sigma_0 = \omega_p^2 / (4\pi\gamma)$ [41] we recover the dielectric permittivity of the Drude model (4). At sufficiently high frequencies $\gamma \ll \omega < \omega_p$ (the region of infrared optics) one can neglect unity as compared to ω/γ and ω^2/γ^2 in (10). Then (4) and (10) lead to the so-called free electron plasma model

$$\varepsilon_p = 1 - \frac{\omega_p^2}{\omega^2}, \quad \sigma(\omega) = \frac{i\sigma_0\gamma}{\omega}. \quad (11)$$

Thus, the plasma model is characterized by pure imaginary conductivity. In the opposite limit $\omega \ll \gamma$ the unity in both numerator and denominator of (10) dominate over ω/γ and ω^2/γ^2 leading to $\sigma(\omega) = \sigma_0$. This converts the dielectric permittivity of the Drude model (4) in the dielectric permittivity of the normal skin effect (9) characterized by real conductivity of conduction electrons σ_0 .

The total current in the framework of the Drude model (4) is given by

$$\begin{aligned} \mathbf{j}_{\text{tot}}(\mathbf{r}, t) &= \text{Re} \left[-\frac{i\omega}{4\pi} \varepsilon_D(\omega) \mathbf{E}_0(\mathbf{r}) e^{-i\omega t} \right] \\ &= \frac{\omega}{4\pi} \left(1 - \frac{\omega_p^2}{\omega^2 + \gamma^2} \right) \text{Im}[\mathbf{E}_0(\mathbf{r}) e^{-i\omega t}] + \frac{\sigma_0\gamma^2}{\omega^2 + \gamma^2} \text{Re}[\mathbf{E}_0(\mathbf{r}) e^{-i\omega t}]. \end{aligned} \quad (12)$$

The first term on the right-hand side of this equation has the meaning of the displacement current, whereas the second term, in accordance to (6), is proportional to the physical electric field $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and describes the real current of conduction electrons. Under the condition $\gamma \ll \omega < \omega_p$, i.e., in the region of infrared optics, the first term dominates. This is the displacement current of the plasma model with a pure imaginary conductivity (11). Under the opposite condition $\omega \ll \gamma$, i.e., in the region of the normal skin effect, the second term on the right-hand side of (12), i.e., the real physical current of conduction electrons dominates.

After the above discussion on the derivation of the Drude model, we now return to the role played by the finite size of the plates. Let us consider plane waves of zero-point oscillations and thermal photons in between the plates incident on their interior boundary surfaces. It is common knowledge (see, e.g., [42, 43]) that charge carriers in a conductor move in response to electromagnetic oscillations. For plates of a finite area, the application condition for the derivation of the Drude model is formally violated because the extension of the oscillation wave fronts is much larger than the size of any conceivable plates. However, if the frequencies of oscillations are high enough (recall that at room temperature the first Matsubara frequency

is equal to $\xi_1 \approx 2.47 \times 10^{14} \text{ rad s}^{-1}$, and all others with $l \geq 1$ are, respectively, higher) there is no accumulation of charges on the side boundary surfaces of finite metal plates. First, at so high frequencies the real current of conduction electrons [given by the second term on the right-hand side of (12)] is small in comparison with the displacement current. Second, a high-frequency electric field quickly changes its direction. As a result, electric charges which are accumulated on the sides of a plate change their sign many times during any reasonable time of force measurement. This leads to practically zero mean surface charge. Thus, equations (4), (7), (8) and (12) remain macroscopically valid.

The situation changes drastically when the contribution from the zero Matsubara frequency $\xi_0 = 0$ is considered. The plane wave of zero frequency should be understood as the limit of plane waves with some low frequencies ξ in the case that $\xi \rightarrow 0$. As was mentioned above, the extension of a wave front far exceeds the size of the plates. If it is remembered that the period of the wave of vanishing frequency goes to infinity, the Casimir plates are found in practically constant electric and magnetic fields. As is described in textbooks on classical electrodynamics (see, e.g., [42, 43]), in a quasistatic case the propagation direction of a plane wave inside a metal is approximately perpendicular to its surface independently of the angle of incidence. Thus, a short-lived current which arises under the influence of a constant electric field in the plane of plates immediately gives rise to some nonzero surface charge densities ρ of opposite signs accumulated on opposite sides of the plates. The electric field generated by these charges precisely compensates the electric field of zero frequency inside a metal. As a result, the electric field inside the metal is exactly equal to zero [42, 43]. As to the space between the plates, the resulting field there is the superposition of an approximately constant field of external sources and of the field generated by the charge distribution on the plate sides.

From what has been said, it appears that Maxwell equation (5) and all consequences obtained from them are not applicable in the case of plane waves of zero frequency. In that case for finite plates not only a nonzero-induced current must be taken into account but also a nonzero-induced charge density generated by this current which, however, is omitted in (5). As was noted by Parsegian (see [37, p 254]), ‘conductors must be considered case by case corresponding to the limitations imposed by boundary surfaces.’ Here we demonstrate that these limitations arise from the substitution of the Drude dielectric function (which is obtained for unbounded medium) in the zero-frequency term of the Lifshitz formula. Such substitution is in contradiction with electrodynamics, because, as was shown above, the Drude model admits a nonzero current of conduction electrons, whereas electrodynamics ascertains that the current of conduction electrons inside a finite metal plate placed in a plane wave of zero frequency must be equal to zero.

We emphasize that the time interval during which charges on the sides of finite plates are accumulated and the total electric field in a metal turns into zero is extremely short. To make sure that this is the case we describe the dynamic process of charge accumulation on the sides of the plates by a simple model:

$$\frac{d\rho(t)}{dt} = \sigma_0 E_{\text{tot}}(t), \quad E_{\text{tot}}(t) = E - E_\rho(t), \quad (13)$$

where E is the constant electric field along the plates and $E_\rho(t)$ is the field produced by the surface charge density till the moment t . For the order of magnitude estimation, it is sufficient to represent $E_\rho(t)$ as the field in a plane capacitor: $E_\rho(t) = 4\pi\rho(t)$. Then we arrive at the following solution:

$$\rho(t) = \frac{E}{4\pi} (1 - e^{-4\pi\sigma_0 t}). \quad (14)$$

As an example, for Au it holds $4\pi\sigma_0 = 3.5 \times 10^{18} \text{ s}^{-1}$ and, thus, even after a very short time lapse of $t = 10^{-18} \text{ s}$, $\rho(t)$ practically achieves the maximum value $\rho(\infty) = E/(4\pi)$. Then

from equation (13) it follows that the total field inside a metal vanishes, $E_{\text{tot}}(\infty) = 0$, as it should be.

Thus, the substitution of the Drude dielectric function in the zero-frequency term of the Lifshitz formula is self-contradictory. As was recalled in the introduction, for the Drude model it holds $r_{\text{TE}}(0, k_{\perp}) = 0$. This means that the TE field of zero frequency completely penetrates into a metal. For metal plates of finite size, this unavoidably leads to instant accumulation of induced charges on the plate sides and vanishing of both an electric field and a current inside a metal. However, the Lifshitz formula is derived for neutral plates without any nonzero surface charge densities. At the same time, the Drude model admits the presence of a nonzero-induced current. Because of this, it is not surprising that the Lifshitz theory in combination with the Drude model violates the Nernst heat theorem for perfect crystal lattices [16, 17] and was found to be in contradiction with several experiments [23–26]. If metal plates were really infinite (as is formally suggested in the derivation of the Lifshitz formula) the Drude model would be applicable including the zero-frequency term. This, however, is an unphysical case and it cannot be considered as a closed system where the laws of thermodynamics must be valid.

The above discussion uses the formulation of the Lifshitz formula (1) in terms of the imaginary frequency axis. However, direct computations using the formulation in terms of real frequencies show [44] that the region of sufficiently low real frequencies results in precisely the same contribution to the Casimir-free energy as does the zero Matsubara frequency. As a result, all above conclusions are equally applicable to the contribution into the free energy from the zero-frequency term in the Matsubara formulation and to the equivalent contribution from low real frequencies in the formalism of real frequency axis.

By contrast with the Drude model, the plasma dielectric function (11) does not lead to a real current of conduction electrons and does not result in accumulation of charges on the side surfaces of finite metal plates. The free electron plasma model in combination with the Lifshitz formula satisfies the Nernst heat theorem [16, 17]. However, as was mentioned in the introduction, it is in disagreement with short separation experiments on the measurement of the Casimir force. Below we demonstrate that the generalized plasma-like permittivity [36], which is in agreement with all experiments performed up to date also satisfies the requirements of thermodynamics. Thus, it is becoming the best-known candidate for the adequate description of metals in the framework of the Lifshitz theory.

3. Thermodynamic test for the generalized plasma-like dielectric permittivity

The generalized plasma-like dielectric permittivity can be presented in the form [36]

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} + A(\omega), \quad (15)$$

where the additional term $A(\omega)$ takes into account the interband transitions of core electrons. Explicitly it is given by

$$A(\omega) = \sum_{j=1}^K \frac{f_j}{\omega_j^2 - \omega^2 - ig_j\omega}, \quad (16)$$

where $\omega_j \neq 0$ are the resonant frequencies of oscillators describing the core electrons, g_j are the respective relaxation parameters, f_j are the oscillator strengths and K is the number of oscillators. The values of oscillator parameters for different materials can be found in [37]. Recently the precise determination of these parameters for Au was performed in [26]. Note that the generalized plasma model does not include relaxation of the free conduction electrons.

The latter are described by an oscillator with zero resonant frequency, $\omega_0 = 0$, which is not contained in (16) but is explicitly included in (15) with $g_0 = 0$ and $f_0 = \omega_p^2$. Thus, similar to the usual plasma model (11), the generalized plasma-like permittivity (15) and (16) admits only the displacement current and does not allow for the accumulation of charges on the sides of finite plates. Because of this, the generalized plasma-like permittivity is compatible with the Lifshitz formula which is derived for neutral plates with zero charge distributions.

In [36], it was shown that the permittivity (15) and (16) precisely satisfies the Kramers–Kronig relations. Here we prove that the Lifshitz formula combined with the generalized plasma-like permittivity is in agreement with the Nernst heat theorem and thus withstands the thermodynamic test.

To find the asymptotic behavior of the Casimir-free energy and entropy at low temperature, we first present equations (1), (2) and (15), (16) in terms of the following dimensionless parameters,

$$\begin{aligned} \tilde{\omega}_p &= \frac{\omega_p}{\omega_c} \equiv \frac{1}{\alpha}, & \zeta_l &= \frac{\xi_l}{\omega_c} \equiv \tau l, & y &= \sqrt{4a^2 k_\perp^2 + \zeta_l^2}, \\ \gamma_j &= \frac{\omega_c^2}{\omega_j^2}, & \delta_j &= \frac{\omega_c g_j}{\omega_j^2}, & C_j &= \frac{f_j}{\omega_j^2}, \end{aligned} \tag{17}$$

where $\omega_c \equiv c/(2a)$ is the so-called characteristic frequency of the Casimir effect. In terms of new variables, the Lifshitz formula (1) takes the form

$$\begin{aligned} \mathcal{F}(a, T) &= \frac{\hbar c \tau}{32\pi^2 a^3} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2} \delta_{0l}\right) \int_{\zeta_l}^{\infty} y \, dy \left\{ \ln [1 - r_{\text{TM}}^2(\zeta_l, y) e^{-y}] \right. \\ &\quad \left. + \ln [1 - r_{\text{TE}}^2(\zeta_l, y) e^{-y}] \right\}. \end{aligned} \tag{18}$$

The reflection coefficients (2) are given by

$$\begin{aligned} r_{\text{TM}}(\zeta_l, y) &= \frac{(\varepsilon_l^2 - 1)(y^2 - \zeta_l^2)}{(\varepsilon_l + 1)y^2 + (\varepsilon_l - 1)\zeta_l^2 + 2\varepsilon_l y h_l(y) \coth\left[\frac{d}{2a} h_l(y)\right]}, \\ r_{\text{TE}}(\zeta_l, y) &= \frac{(\varepsilon_l - 1)\zeta_l^2}{2y^2 + (\varepsilon_l - 1)\zeta_l^2 + 2y h_l(y) \coth\left[\frac{d}{2a} h_l(y)\right]}, \end{aligned} \tag{19}$$

where

$$h_l(y) = [y^2 + (\varepsilon_l - 1)\zeta_l^2]^{1/2}. \tag{20}$$

The generalized plasma-like dielectric permittivity along the imaginary frequency axis can be presented as

$$\varepsilon_l = \varepsilon(i\zeta_l) = 1 + \frac{\tilde{\omega}_p^2}{\zeta_l^2} + A_l = 1 + \frac{1}{\alpha^2 \zeta_l^2} + A_l, \tag{21}$$

where

$$A_l = A(\zeta_l) = \sum_{j=1}^K \frac{C_j}{1 + \gamma_j \zeta_l^2 + \delta_j \zeta_l}. \tag{22}$$

Using the Abel–Plana formula [3, 5]

$$\sum_{l=0}^{\infty} \left(1 - \frac{1}{2} \delta_{0l}\right) F(l) = \int_0^{\infty} F(t) \, dt + i \int_0^{\infty} dt \frac{F(it) - F(-it)}{e^{2\pi t} - 1}, \tag{23}$$

where $F(z)$ is an analytic function in the right half of the complex plane, we can rearrange (18) to the form

$$\mathcal{F}(a, T) = E(a) + \Delta\mathcal{F}(a, T). \quad (24)$$

Here, the energy of the Casimir interaction at zero temperature is given by

$$E(a) = \frac{\hbar c}{32\pi^2 a^3} \int_0^\infty d\zeta \int_\zeta^\infty f(\zeta, y) dy \quad (25)$$

and the function $f(\zeta, y)$ is defined as

$$f(\zeta, y) = y \ln [1 - r_{\text{TM}}^2(\zeta, y) e^{-y}] + y \ln [1 - r_{\text{TE}}^2(\zeta, y) e^{-y}]. \quad (26)$$

The thermal correction to the Casimir energy is expressed as follows:

$$\Delta\mathcal{F}(a, T) = \frac{i\hbar c \tau}{32\pi^2 a^3} \int_0^\infty dt \frac{F(i\tau t) - F(-i\tau t)}{e^{2\pi t} - 1}, \quad (27)$$

where

$$F(x) = \int_x^\infty dy f(x, y). \quad (28)$$

The behavior of the thermal correction (27) and (28) at low temperature will be the subject of our further consideration.

Perturbation expansion can be performed in analogy to papers [16, 17, 45, 46]. At first, we expand the reflection coefficients (19) with ζ_l replaced by ζ in powers of parameter α defined in (17) preserving all powers up to the fourth inclusive. The parameter α can be identically presented as $\alpha = \lambda_p / (4\pi a)$, where λ_p is the plasma wavelength. This means that $\alpha \ll 1$ at all separation distances between the plates larger than λ_p . As is seen from (26), it is more convenient to expand the logarithmic functions containing the reflection coefficients in (26) multiplied by the variable y . The results are as follows:

$$\begin{aligned} y \ln [1 - r_{\text{TM}}^2(\zeta, y) e^{-y}] &= y \ln(1 - e^{-y}) + \alpha \frac{4\zeta^2}{e^y - 1} - \alpha^2 \frac{8e^y \zeta^4}{y(e^y - 1)^2} \\ &+ \alpha^3 \frac{2\zeta^2 \{2\zeta^4 (3e^y + 1)^2 + 3(e^y - 1)^2 y^2 [y^2 - 2\zeta^2 - \zeta^2 A(\zeta)]\}}{3y^2 (e^y - 1)^3} \\ &- \alpha^4 \frac{8e^y \zeta^4 \{2\zeta^4 (e^y + 1)^2 + (e^y - 1)^2 y^2 [y^2 - 2\zeta^2 - \zeta^2 A(\zeta)]\}}{y^3 (e^y - 1)^4}, \\ y \ln [1 - r_{\text{TE}}^2(\zeta, y) e^{-y}] &= y \ln(1 - e^{-y}) + \alpha \frac{4y^2}{e^y - 1} - \alpha^2 \frac{8y^3 e^y}{(e^y - 1)^2} \\ &+ \alpha^3 \frac{2y^2 [-3(e^y - 1)^2 \zeta^2 A(\zeta) + y^2 (15e^{2y} + 18e^y - 1)]}{3(e^y - 1)^3} \\ &- \alpha^4 \frac{8y^3 e^y [-(e^y - 1)^2 \zeta^2 A(\zeta) + y^2 (e^{2y} + 6e^y + 1)]}{(e^y - 1)^4}. \end{aligned} \quad (29)$$

It is significant that these expansions do not depend on d (the thickness of the plates) contained in (19). This is because the factor in the denominator of (19),

$$\begin{aligned} \coth \left[\frac{d}{2a} h_l(y) \right] &= \coth \left(\frac{d}{2a} \sqrt{y^2 + \frac{1}{\alpha^2} + A_l \zeta_l^2} \right) \\ &= \frac{1 + \exp \left(-\frac{d}{a\alpha} \sqrt{1 + \alpha^2 y^2 + \alpha^2 A_l \zeta_l^2} \right)}{1 - \exp \left(-\frac{d}{a\alpha} \sqrt{1 + \alpha^2 y^2 + \alpha^2 A_l \zeta_l^2} \right)}, \end{aligned} \quad (30)$$

behaves asymptotically as

$$1 + 2 \exp\left(-\frac{d}{a\alpha}\right) + \dots \tag{31}$$

when α goes to zero. Thus, this factor could only contribute exponentially small terms in the expansion (29) providing the plate thickness d is much larger than the penetration depth of electromagnetic oscillations into the metal [recall that $2a\alpha = \lambda_p/(2\pi)$]. Under this condition, the perturbation expansions (29) are common for two semispaces and for two plates of finite thickness. We note also that terms in (29) of order α^0, α and α^2 do not contain contributions from the core electrons. They are the same as for the usual free electron plasma model (11). The contributions from the core electrons are contained only in the terms of order α^3 and α^4 in (29).

The parameter τ defined in (17) can be identically represented as

$$\tau = 2\pi \frac{T}{T_{\text{eff}}}, \quad k_B T_{\text{eff}} \equiv \frac{\hbar c}{2a}. \tag{32}$$

Here T_{eff} is the so-called effective temperature. For example, at a separation distance of $a = 1\mu$ m $T_{\text{eff}} \approx 1145$ K. Below we will consider the limiting case of low temperatures $T \ll T_{\text{eff}}$. The contribution from the terms of order α^0, α and α^2 in (29) into the thermal correction (26) was found in [45, 46] where the usual free electron plasma model was considered. This contribution is given by

$$\Delta\mathcal{F}_p(a, T) = -\frac{\hbar c}{32\pi^2 a^3} \left\{ \frac{\zeta(3)}{4\pi^2} \tau^3 - \frac{1}{360} \tau^4 + \alpha \left[\frac{\zeta(3)}{\pi^2} \tau^3 - \frac{1}{45} \tau^4 \right] - \alpha^2 \frac{6\zeta(5)}{\pi^2} \tau^5 \right\},$$

where $\zeta(z)$ is the Riemann zeta function. As was shown in [46], the terms in (33) of order α^0 and α do not contain corrections of order τ^n with $n \geq 5$. They contain only the exponentially small corrections of order $\exp(-2\pi/\tau)$.

Now we deal with the terms of order α^3 and α^4 in (29) which contain the contributions from the core electrons. From (26), (28) and (29) the respective functions $F^{(3)}(x)$ and $F^{(4)}(x)$ are given by

$$\begin{aligned} F^{(3)}(x) = & -2\alpha^3 \left\{ [A(x) - 1]x^2 \int_x^\infty \frac{y^2 dy}{e^y - 1} - \frac{1}{3} \int_x^\infty \frac{y^4(15e^{2y} + 18e^y - 1)}{(e^y - 1)^3} \right. \\ & \left. + [A(x) + 2]x^4 \int_x^\infty \frac{dy}{e^y - 1} - \frac{2}{3}x^6 \int_x^\infty \frac{(3e^y + 1)^2}{y^3(e^y - 1)^3} dy \right\}, \\ F^{(4)}(x) = & 8\alpha^4 \left\{ A(x)x^2 \int_x^\infty \frac{y^3 e^y dy}{(e^y - 1)^2} - \int_x^\infty \frac{y^5(e^{2y} + 6e^y + 1)e^y dy}{(e^y - 1)^4} \right. \\ & \left. + [A(x) + 2]x^6 \int_x^\infty \frac{e^y dy}{y(e^y - 1)^2} - x^4 \int_x^\infty \frac{ye^y dy}{y^3(e^y - 1)^2} \right. \\ & \left. - 2x^8 \int_x^\infty \frac{e^y(e^y + 1)^2}{y^3(e^y - 1)^4} dy \right\}. \end{aligned} \tag{34}$$

Calculating all integrals in (34) as asymptotic expansions at small x (see the appendix for details) we arrive at

$$\begin{aligned}
 F^{(3)}(i\tau t) - F^{(3)}(-i\tau t) &= -2i\alpha^3 \left[2\tau^3 t^3 \zeta(3) \sum_{j=1}^K C_j \delta_j + \pi \tau^4 t^4 \left(\sum_{j=1}^K C_j + 2 \right) \right], \\
 F^{(4)}(i\tau t) - F^{(4)}(-i\tau t) &= 8i\alpha^4 \left[8\tau^3 t^3 \zeta(3) \sum_{j=1}^K C_j \delta_j + \pi \tau^4 t^4 \right].
 \end{aligned} \tag{35}$$

The terms omitted in (35) are of order τ^5 .

Substituting (35) in (27) and integrating with respect to t , we obtain the contribution to the thermal correction from the terms of order α^3 and α^4 :

$$\begin{aligned}
 \Delta \mathcal{F}_g(a, T) &= -\frac{\hbar c}{32\pi^2 a^3} \left\{ -\alpha^3 \left[\frac{\zeta(3)}{60} \sum_{j=1}^K C_j \delta_j \tau^4 + \frac{3\zeta(5)}{2\pi^4} \left(\sum_{j=1}^K C_j + 2 \right) \tau^5 \right] \right. \\
 &\quad \left. + \alpha^4 \left[\frac{4\zeta(3)}{15} \sum_{j=1}^K C_j \delta_j \tau^4 + \frac{6\zeta(5)}{\pi^4} \tau^5 \right] \right\}.
 \end{aligned} \tag{36}$$

Total Casimir's free energy computed using the generalized plasma-like permittivity can be now found from (24), (33) and (36):

$$\mathcal{F}(a, T) = E(a) + \Delta \mathcal{F}_p(a, T) + \Delta \mathcal{F}_g(a, T). \tag{37}$$

Taking into account (32), it can be represented in the form

$$\begin{aligned}
 \mathcal{F}(a, T) &= E(a) - \frac{\hbar c \zeta(3)}{16\pi a^3} \left(\frac{T}{T_{\text{eff}}} \right)^3 \\
 &\quad \times \left\{ 1 + 4\alpha - \frac{\pi^3}{45\zeta(3)} \frac{T}{T_{\text{eff}}} \left(1 + 8\alpha + 6\zeta(3)\alpha^3 \sum_{j=1}^K C_j \delta_j - 96\zeta(3)\alpha^4 \sum_{j=1}^K C_j \delta_j \right) \right. \\
 &\quad \left. - \frac{96\pi^2 \zeta(5)}{\zeta(3)} \left(\frac{T}{T_{\text{eff}}} \right)^2 \alpha^2 \left[1 + \frac{\alpha}{4\pi^2} \left(\sum_{j=1}^K C_j + 2 \right) - \frac{\alpha^2}{\pi^2} \right] \right\}.
 \end{aligned} \tag{38}$$

Here one can see that the free energy calculated using the generalized plasma-like permittivity contains the correction of order $(T/T_{\text{eff}})^4$ not only in the terms of order α^0 and α (as in the usual plasma model) but also in the third- and fourth-order expansion terms in α . In the usual plasma model, the terms of order α^3 and α^4 contain the thermal corrections only of order of $(T/T_{\text{eff}})^5$ and higher [45]. To estimate the relative role of the additional terms arising due to the use of the generalized plasma-like permittivity, one can use the parameters of oscillator terms in (22) for Au [26]. This results in

$$\sum_{j=1}^6 C_j = 6.3175, \quad \sum_{j=1}^6 C_j \delta_j = \begin{cases} 0.272, & a = 200 \text{ nm}, \\ 0.109, & a = 500 \text{ nm}. \end{cases} \tag{39}$$

From (38) it is easy to find the asymptotic behavior of the Casimir entropy

$$S(a, T) = -\frac{\partial \mathcal{F}(a, T)}{\partial T} \tag{40}$$

at low temperature. The result is

$$\begin{aligned}
 S(a, T) = & \frac{3\zeta(3)k_B}{8\pi a^2} \left(\frac{T}{T_{\text{eff}}} \right)^2 \\
 & \times \left\{ 1 + 4\alpha - \frac{4\pi^3}{135\zeta(3)} \frac{T}{T_{\text{eff}}} \left(1 + 8\alpha + 6\zeta(3)\alpha^3 \sum_{j=1}^K C_j \delta_j - 96\zeta(3)\alpha^4 \sum_{j=1}^K C_j \delta_j \right) \right. \\
 & \left. - \frac{160\pi^2\zeta(5)}{\zeta(3)} \left(\frac{T}{T_{\text{eff}}} \right)^2 \alpha^2 \left[1 + \frac{\alpha}{4\pi^2} \left(\sum_{j=1}^K C_j + 2 \right) - \frac{\alpha^2}{\pi^2} \right] \right\}. \quad (41)
 \end{aligned}$$

As is seen from (41),

$$S(a, T) \rightarrow 0 \quad \text{when } T \rightarrow 0, \quad (42)$$

i.e., the entropy goes to zero (and remains positive) when the temperature vanishes. This means that the Nernst heat theorem is satisfied and the Lifshitz theory combined with the generalized plasma-like dielectric permittivity withstands the thermodynamic test.

4. Conclusions and discussion

In the foregoing we have continued the elaboration of new theoretical approach to the thermal Casimir force based on the Lifshitz formula combined with the generalized plasma-like dielectric permittivity [36]. In the first part of the paper (section 2) the physical reasons were presented why the Drude dielectric function is not applicable in the case of finite metallic plates. It was shown that for the validity of the Drude model the nonzero current of conduction electrons must exist, whereas the surface charge densities must be equal to zero. Both these conditions are shown to be violated when the plane wave of electromagnetic oscillations of vanishing frequency falls on a finite metal plate. In this case the electric field and current of conduction electrons practically instantaneously turn into zero. This is accompanied by the accumulation of charges on the sides of plates. Not only the Drude dielectric function, but also the Lifshitz formula are not applicable to this physical situation. In contrast, the generalized plasma-like permittivity leads to only a displacement current and does not result in the accumulation of surface charges. The obtained results furnish insights into the long-debated problem why the Lifshitz theory combined with the Drude dielectric function results in contradictions with thermodynamics and experiment. It becomes clear also why the generalized plasma-like permittivity, which does not include the relaxation processes of conduction electrons, is consistent with all available measurements of the Casimir force at both short and large separations.

Recent paper [47] also argues that the finite size effects of the conductors may play an important role in the problem of the thermal Casimir effect. This was illustrated in the simplified case of two wires of finite length described by the Drude model and interacting through the inductive coupling between Johnson currents. If the capacitive effects associated with the end points of the wires are not taken into account, the thermal interaction between the wires leads to the violation of the Nernst heat theorem. If the capacitive effects were taken into consideration, the agreement with thermodynamics is restored [47].

To conclusively establish the applicability of the generalized plasma-like permittivity in the theory of the thermal Casimir force between metals, in section 3 we have performed the thermodynamic test of this model. We have analytically found the asymptotic behavior of both the Casimir-free energy and Casimir entropy at low temperature. This was done using the perturbation theory in two small parameters. The obtained new expressions generalize the

previously known ones (found for the usual free electron plasma model which does not take dissipation into account). When the oscillator parameters describing the core electrons go to zero, the newly obtained expressions for the Casimir's free energy and entropy go into those found for the usual plasma model. The Casimir entropy at low temperature derived using the generalized plasma-like permittivity is positive and takes zero value at zero temperature. Thus, the Nernst heat theorem is satisfied.

To conclude, the generalized plasma-like permittivity provides a good basis in agreement with thermodynamics and experiment for the description of the thermal Casimir force between metallic plates of finite size using the standard Lifshitz theory. A more fundamental approach to the resolution of this problem would require, in accordance with Parsegian's insight [37], the consideration from the very beginning of finite plates and charging of their boundary surfaces. This, however, goes far beyond the scope of the Lifshitz theory.

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Appendix

Here we derive equation (35), where functions $F^{(3)}(x)$ and $F^{(4)}(x)$ are defined in (34).

The first integral in the definition of $F^{(3)}(x)$ converges when $x \rightarrow 0$ and can be calculated as

$$\begin{aligned} I_1^{(3)}(x) &\equiv \int_x^\infty \frac{y^2 dy}{e^y - 1} = 2\text{Li}_3(e^{-x}) + 2x\text{Li}_2(e^{-x}) - x^2 \ln(1 - e^{-x}) \\ &= 2\zeta(3) - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{48} + O(x^6), \end{aligned} \quad (\text{A.1})$$

where $\text{Li}_n(z)$ is the polylogarithm function [48]. Expanding the function $A(x)$ defined in (22) in powers of x and using (A.1) we arrive at

$$\begin{aligned} [A(x) - 1]x^2 I_1^{(3)}(x) &= -2\zeta(3)x^2 + \frac{x^4}{2} + 2\zeta(3) \sum_{j=1}^K C_j x^2 - 2\zeta(3) \sum_{j=1}^K C_j \delta_j x^3 \\ &\quad - \left[\frac{1}{2} \sum_{j=1}^K C_j - 2\zeta(3) \sum_{j=1}^K C_j \delta_j^2 + 2\zeta(3) \sum_{j=1}^K C_j \gamma_j \right] x^4 + O(x^5), \end{aligned} \quad (\text{A.2})$$

From (A.2), only the term proportional to x^3 contributes to (35).

The second integral in the definition of $F^{(3)}(x)$ in (34) also converges when $x \rightarrow 0$. It can be found in the form

$$\begin{aligned} I_2^{(3)}(x) &\equiv \int_x^\infty \frac{y^4(15e^{2y} + 18e^y - 1)dy}{(e^y - 1)^3} = 15 \sum_{n=1}^\infty n^2 \int_x^\infty y^4 e^{-ny} dy \\ &\quad + \int_x^\infty \frac{y^4 e^{-2y} dy}{(1 - e^{-y})^2} = -16 \frac{x^2}{2} + \frac{x^4}{5} + O(x^5). \end{aligned} \quad (\text{A.3})$$

This does not contribute to (35) in the perturbation orders under consideration.

The third integral in $F^{(3)}(x)$,

$$I_3^{(3)}(x) \equiv \int_x^\infty \frac{e^{-y} dy}{1 - e^{-y}} = -\ln(1 - e^{-x}), \tag{A.4}$$

diverges when x goes to zero. It should be, however, multiplied by $[A(x) + 2]x^4$ with the result

$$[A(x) + 2]x^4 I_3^{(3)}(x) = - \left(\sum_{j=1}^K C_j + 2 \right) x^4 \ln x + \sum_{j=1}^K C_j \delta_j x^5 \ln x + O(x^5). \tag{A.5}$$

Only the first term on the right-hand side of (A.5) contributes to (35). This contribution is simply found when taken into account that

$$\ln(iz) - \ln(-iz) = i\pi. \tag{A.6}$$

The last, fourth, integral in the definition of $F^{(3)}(x)$, also diverges when x goes to zero. It can be identically represented in the form

$$\begin{aligned} I_4^{(3)}(x) \equiv & \int_x^\infty \frac{e^{-y}(3 + e^{-y})^2}{y^2(1 - e^{-y})^3} dy = 16 \int_x^\infty \frac{e^{-y}}{y^5} dy + 16 \int_x^\infty \frac{e^{-y}}{y^4} dy \\ & + 9 \int_x^\infty \frac{e^{-y}}{y^3} dy + \frac{19}{6} \int_x^\infty \frac{e^{-y}}{y^2} dy + \frac{41}{60} \int_x^\infty \frac{e^{-y}}{y} dy \\ & + \int_x^\infty \frac{e^{-y}}{y^2} \left[\frac{(3 + e^{-y})^2}{(1 - e^{-y})^3} - \frac{16}{y^3} - \frac{16}{y^2} - \frac{9}{y} - \frac{19}{6} - \frac{41y}{60} \right] dy. \end{aligned} \tag{A.7}$$

The last integral on the right-hand side of (A.7) converges when $x \rightarrow 0$. It does not contribute to the perturbation orders of our interest after the multiplication by x^5 . As a result, we arrive at

$$\begin{aligned} x^6 I_4^{(3)}(x) = & x^6 \left[16\Gamma(-4, x) + 16\Gamma(-3, x) + 9\Gamma(-2, x) \right. \\ & \left. + \frac{19}{6}\Gamma(-1, x) + \frac{41}{60}\Gamma(-0, x) \right] + O(x^6), \end{aligned} \tag{A.8}$$

where $\Gamma(n, x)$ is the incomplete gamma function [49]. Using the identity [49]

$$\Gamma(-n, x) = \frac{(-1)^n}{n!} \left[\Gamma(0, x) - e^{-x} \sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^{m+1}} \right], \tag{A.9}$$

where $n = 1, 2, \dots$, and the asymptotic relation

$$\Gamma(0, x) = -\gamma - \ln x + x - \frac{x^2}{4} + \frac{x^3}{18} + O(x^4) \tag{A.10}$$

with Euler’s constant $\gamma = 0.577216$, we finally obtain

$$x^6 I_4^{(3)}(x) = 4x^2 + \frac{x^4}{2} + O(x^5). \tag{A.11}$$

This evidently does not contribute to (35).

As a result, by using (A.2) and (A.5) we obtain the first equation in (35).

Now we consider the derivation of the second equation in (35) containing the function $F^{(4)}(x)$ defined in (34). The first integral in $F^{(4)}(x)$,

$$I_1^{(4)}(x) \equiv \int_x^\infty \frac{y^3 e^{-y}}{(1 - e^{-y})^2} dy, \tag{A.12}$$

converges when x goes to zero. It is calculated in analogy to (A.1). As a result the following expansion is obtained:

$$A(x)x^2I_1^{(4)}(x) = 6\zeta(3) \sum_{j=1}^K C_j x^2 - 6\zeta(3) \sum_{j=1}^K C_j \delta_j x^3 - \left[\frac{1}{2} \sum_{j=1}^K C_j - 6\zeta(3) \sum_{j=1}^K C_j \delta_j^2 + 6\zeta(3) \sum_{j=1}^K C_j \gamma_j \right] x^4 + O(x^5). \quad (\text{A.13})$$

The second term on the right-hand side of (A.13) contributes to (35).

The second integral in $F^{(4)}(x)$ also converges when x goes to zero. It can be calculated similar to $I_2^{(3)}(x)$ in (A.2) and does not contain odd powers of x lower than x^5 . Thus, this integral does not contribute to (35).

The third and fifth integrals in the definition of $F^{(4)}(x)$, (34), diverge in the limit $x \rightarrow 0$. However, by calculating them similar to $I_4^{(3)}(x)$ in (A.7)–(A.10) and multiplying the results by x^6 and x^8 , respectively, we find that both these integrals do not contribute to (35).

The fourth integral in $F^{(4)}(x)$ can be calculated as follows:

$$I_4^{(4)}(x) \equiv \int_x^\infty \frac{y e^{-y} dy}{(1 - e^{-y})^2} = -\ln(1 - e^{-x}) + \frac{x e^{-x}}{1 - e^{-x}}. \quad (\text{A.14})$$

It diverges when x goes to zero. After multiplication by x^4 one obtains

$$x^4 I_4^{(4)}(x) = x^2 - x^4 \ln x + O(x^6). \quad (\text{A.15})$$

The second term on the right-hand side of (A.15) contributes to (35).

Using (A.13) and (A.15) we obtain the second equation in (35).

References

- [1] Casimir H B G 1948 *Proc. K. Ned. Akad. Wet.* **51** 793
- [2] Jaffe R L 2005 *Phys. Rev. D* **72** 021301
- [3] Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1
- [4] Milonni P W 1994 *The Quantum Vacuum* (New York: Academic)
- [5] Mostepanenko V M and Trunov N N 1997 *The Casimir Effect and its Applications* (Oxford: Clarendon)
- [6] Milton K A 2001 *The Casimir Effect* (Singapore: World Scientific)
- [7] Plunien G, Müller B and Greiner W 1986 *Phys. Rep.* **134** 87
- [8] Lamoreaux S K 2005 *Rep. Prog. Phys.* **68** 201
- [9] Elizalde E and Odintsov S D (ed) 2006 Papers presented at the *7th Workshop on Quantum Field Theory Under the Influence of External Conditions* *J. Phys. A: Math. Gen.* **39** 6109
- [10] Emig T, Jaffe R L, Kardar M and Scardicchio A 2006 *Phys. Rev. Lett.* **96** 080403
- [11] Bordag M 2006 *Phys. Rev. D* **73** 125018
- [12] Chan H B, Aksyuk V A, Kleiman R N, Bishop D J and Capasso F 2001 *Science* **291** 1941
Chan H B, Aksyuk V A, Kleiman R N, Bishop D J and Capasso F 2001 *Phys. Rev. Lett.* **87** 211801
- [13] Emig T 2007 *Phys. Rev. Lett.* **98** 160801
- [14] Ashourvan A, Miri M and Golestanian R 2007 *Phys. Rev. Lett.* **98** 140801
Ashourvan A, Miri M and Golestanian R 2007 *Phys. Rev. E* **75** 040103
- [15] Lifshitz E M 1956 *Sov. Phys.—JETP* **2** 73
- [16] Bezerra V B, Klimchitskaya G L and Mostepanenko V M 2002 *Phys. Rev. A* **66** 062112
- [17] Bezerra V B, Klimchitskaya G L, Mostepanenko V M and Romero C 2004 *Phys. Rev. A* **69** 022119
- [18] Brevik I, Aarseth J B, Høye J S and Milton K A 2005 *Phys. Rev. E* **71** 056101
- [19] Bezerra V B, Decca R S, Fischbach E, Geyer B, Klimchitskaya G L, Krause D E, López D, Mostepanenko V M and Romero C 2006 *Phys. Rev. E* **73** 028101
- [20] Mostepanenko V M, Bezerra V B, Decca R S, Fischbach E, Geyer B, Klimchitskaya G L, Krause D E, López D and Romero C 2006 *J. Phys. A: Math. Gen.* **39** 6589
- [21] Milton K A 2004 *J. Phys. A: Math. Gen.* **37** R209

- [22] Høyve J S, Brevik I, Aarseth J B and Milton K A 2006 *J. Phys. A: Math. Gen.* **39** 6031
- [23] Decca R S, Fischbach E, Klimchitskaya G L, Krause D E, López D and Mostepanenko V M 2003 *Phys. Rev. D* **68** 116003
- [24] Decca R S, López D, Fischbach E, Klimchitskaya G L, Krause D E and Mostepanenko V M 2005 *Ann. Phys., NY* **318** 37
- [25] Decca R S, López D, Fischbach E, Klimchitskaya G L, Krause D E and Mostepanenko V M 2007 *Phys. Rev. D* **75** 077101
- [26] Decca R S, López D, Fischbach E, Klimchitskaya G L, Krause D E and Mostepanenko V M 2007 *Eur. Phys. J. C* **51** 963
- [27] Geyer B, Klimchitskaya G L and Mostepanenko V M 2005 *Phys. Rev. D* **72** 085009
- [28] Geyer B, Klimchitskaya G L and Mostepanenko V M 2006 *Int. J. Mod. Phys. A* **21** 5007
- [29] Klimchitskaya G L, Geyer B and Mostepanenko V M 2006 *J. Phys. A: Math. Gen.* **39** 6495
- [30] Geyer B, Klimchitskaya G L and Mostepanenko V M 2007 *Ann. Phys., NY* DOI:10.1016/j.aop.2007.04.005
- [31] Chen F, Klimchitskaya G L, Mostepanenko V M and Mohideen U 2007 *Opt. Exp.* **15** 4823
- [32] Chen F, Klimchitskaya G L, Mostepanenko V M and Mohideen U 2007 *Phys. Rev. B* **76** 035338
- [33] Harris B W, Chen F and Mohideen U 2000 *Phys. Rev. A* **62** 052109
- [34] Chen F, Klimchitskaya G L, Mohideen U and Mostepanenko V M 2004 *Phys. Rev. A* **69** 022117
- [35] Geyer B, Klimchitskaya G L and Mostepanenko V M 2003 *Phys. Rev. A* **67** 062102
- [36] Klimchitskaya G L, Mohideen U and Mostepanenko V M 2007 *J. Phys. A: Math. Theor.* **40** F339
- [37] Parsegian V A 2005 *Van der Waals forces: A Handbook for Biologists, Chemists, Engineers, and Physicists* (Cambridge: Cambridge University Press)
- [38] Blagov E V, Klimchitskaya G L and Mostepanenko V M 2005 *Phys. Rev. B* **71** 235401
- [39] Gies H and Klingmüller K 2006 *Phys. Rev. Lett.* **97** 220405
- [40] Boström M and Sernelius B E 2000 *Phys. Rev. Lett.* **84** 4757
- [41] Ashcroft N W and Mermin N D 1976 *Solid State Physics* (Philadelphia, PA: Saunders)
- [42] Landau L D, Lifshitz E M and Pitaevskii L P 1984 *Electrodynamics of Continuous Media* (Oxford: Pergamon)
- [43] Jackson J D 1999 *Classical Electrodynamics* (New York: Wiley)
- [44] Bezerra V B, Bimonte G, Klimchitskaya G L, Mostepanenko V M and Romero C 2007 *Eur. Phys. J. C*, DOI:10.1140/epjc/s10052-007-0400-x
- [45] Bordag M, Geyer B, Klimchitskaya G L and Mostepanenko V M 2000 *Phys. Rev. Lett.* **85** 503
- [46] Geyer B, Klimchitskaya G L and Mostepanenko V M 2001 *Int. J. Mod. Phys. A* **16** 3291
- [47] Bimonte G 2007 *New J. Phys.* **9** 281
- [48] Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1981 *Higher Transcendental Functions* vol 1 (New York: Krieger)
- [49] Gradshteyn I A and Ryzhik I M 1980 *Table of Integrals, Series and Products* (New York: Academic)